Midterm Exam Calculus 2

19 March 2018, 9:00-11:00

university of groningen

The exam consists of 4 problems. You have 120 minutes to answer the questions. You can achieve 100 points which includes a bonus of 10 points.

1. **[5+10+5=20 Points]** Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as
$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$
   (a) Show that f is not continuous at $(x,y) = (0,0)$. [Hint: consider f along the curve parametrized as $(x(t), y(t)) = (t, t^2)$.]

   (b) Show that the directional derivatives of f at $(x,y) = (0,0)$ exist in all directions $\boldsymbol{u} = (v,w) \in \mathbb{R}^2$ with $v^2 + w^2 = 1$ by using the definition of directional derivatives. Note that you will have to distinguish between the two cases $w = 0$ and $w \neq 0$.

   (c) Is f differentiable at $(x,y) = (0,0)$? Justify your answer.

2. **[15+10=25 Points]** Let $a, b > 0$. Consider the curve parametrized by $\mathbf{r} : [0,1] \to \mathbb{R}^3$ with
$$\mathbf{r}(t) = \mathrm{e}^{at}\cos(bt)\,\mathbf{i} + \mathrm{e}^{at}\sin(bt)\,\mathbf{j} + \mathrm{e}^{at}\,\mathbf{k}.$$
   (a) Determine the length of the curve and its parametrization by arclength.

   (b) At each point on the curve, determine the curvature of the curve.

3. **[5+10+10=25 Points]** Consider the ellipsoid
$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 3$$
which contains the point $(x_0, y_0, z_0) = (1, 2, 3)$.

   (a) Determine the tangent plane of the ellipsoid at the point (x_0, y_0, z_0).

   (b) Show that near the point (x_0, y_0, z_0) the ellipsoid is locally given as the graph of a function over the (x,y) plane, i.e. there is a function $f : (x,y) \mapsto z$ such that near (x_0, y_0, z_0) the ellipsoid is locally given by $z = f(x,y)$. Compute the partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ and show that the graph of the linearization of f at (x_0, y_0) agrees with the tangent plane found in part (a).

   (c) Use the method of Lagrange multipliers to find the points closest to and farthest away from the origin.

4. **[20 Points]** Evaluate the triple integral
$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{0}^{4-x^2-y^2} \mathrm{e}^{x^2+y^2+z}\, dz\,dx\,dy$$
by using cylinder coordinates.

① a) $f(x(t), y(t)) = \begin{cases} \frac{(t)^2 \cdot t^2}{(t)^4 + (t^2)^2} = \frac{t^4}{2t^4} = \frac{1}{2} & \text{if } (t, t^2) \neq (0,0) \\ 0 & \text{if } (t, t^2) = (0,0) \end{cases}$

Thus, $\lim_{(t,t^2) \to (0,0)} f(x(t), y(t)) = \lim_{t \to 0} \frac{t^2 \cdot t^2}{t^4 + t^4} = \lim_{t \to 0} \frac{t^4}{2t^4} = \frac{1}{2}$

$\neq f(x(0), y(0)) = 0$

5/5

Since $\lim_{t \to 0} f(x(t), y(t)) \neq f(x(0), y(0))$ f is <u>not</u> continuous.

b) $D_u f(0,0) =$

$\lim_{h \to 0} \frac{f(0+hv, 0+hw) - f(0,0)}{h} = \lim_{h \to 0} \frac{f(hv, hw)}{h} = \lim_{h \to 0} \frac{h^2 v^2 hw}{h^4 v^4 + h^2 w^2} \cdot \frac{1}{h}$

$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{h^3 v^2 w}{h^2(v^4 h^2 + w^2)} = \lim_{h \to 0} \frac{v^2 w}{v^4 h^2 + w^2}$

if $w = 0 \rightarrow \lim_{h \to 0} \frac{v^2 \cdot 0}{v^4 h^2 + 0^2} = \lim_{h \to 0} \frac{0}{v^4 h^2} = 0$ $(= D_u f(0,0))$

10/10

if $w \neq 0 \rightarrow \lim_{h \to 0} \frac{v^2 w}{v^4 h^2 + w^2} = \frac{v^2 w}{w^2} = \frac{v^2}{w}$ $(= D_u f(0,0))$

Thus, the directional derivative, $D_u f(0,0)$, of f at $(x,y) = (0,0)$ (is defined) exists for all $u = (v, w)$ and thus exists in all directions $u = (v, w)$.

c) at a) I proved that f is <u>not</u> continuous at $(x,y) = (0,0)$ this means that f also is <u>not</u> differentiable at $(x,y) = (0,0)$

5/5

② a) $S = \int_0^t |r'(\tau)|\, d\tau$

15

$r'(t) = \langle ae^{at}\cos(bt) - be^{at}\sin(bt),\ ae^{at}\sin(bt) + be^{at}\cos(bt),\ ae^{at} \rangle$

$|r'(t)| = \sqrt{(ae^{at}\cos(bt) - be^{at}\sin(bt))^2 + (ae^{at}\sin(bt) + be^{at}\cos(bt))^2 + (ae^{at})^2}$

$= (a^2e^{2at}\cos^2(bt) + b^2e^{2at}\sin^2(bt) - 2abe^{2at}\cos(bt)\sin(bt) + a^2e^{2at}\sin^2(bt)$
$+ b^2e^{2at}\cos^2(bt) + 2abe^{2at}\sin(bt)\cos(bt) + a^2e^{2at})^{0.5}$

$= (a^2e^{2at}(\cos^2(bt) + \sin^2(bt)) + b^2e^{2at}(\sin^2(bt) + \cos^2(bt)) + a^2e^{2at})^{0.5}$

$= (a^2e^{2at} + a^2e^{2at} + b^2e^{2at})^{0.5}$

$= \sqrt{2a^2e^{2at} + b^2e^{2at}}$ $\cancel{= \sqrt{2}ae^{at} + be^{at}}$

$= \sqrt{(2a^2+b^2)e^{2at}} = e^{at}\sqrt{2a^2+b^2}$

$S = S(t) = \int_0^t e^{a\tau}\sqrt{2a^2+b^2}\, d\tau = \sqrt{2a^2+b^2}\int_0^t e^{a\tau}\, d\tau$

$= \sqrt{2a^2+b^2} \cdot \left[\frac{1}{a}e^{a\tau}\right]_0^t = \sqrt{2a^2+b^2} \cdot \frac{1}{a}(e^{at} - 1)$

$S = \sqrt{2a^2+b^2} \cdot \frac{1}{a} \cdot (e^{at} - 1) \Rightarrow \frac{a \cdot s}{\sqrt{2a^2+b^2}} = e^{at} - 1 \Rightarrow$

$\frac{as}{\sqrt{2a^2+b^2}} + 1 = e^{at} \Rightarrow \ln\left(\frac{as}{\sqrt{2a^2+b^2}} + 1\right) = at \Rightarrow$

$t = \frac{1}{a}\ln\left(\frac{as}{\sqrt{2a^2+b^2}} + 1\right) = t(s)$

$r(t(s)) = \left(\frac{a \cdot s}{\sqrt{2a^2+b^2}} + 1\right) \cdot \left\langle \cos\left(\frac{b}{a}\ln\left(\frac{as}{\sqrt{2a^2+b^2}} + 1\right)\right), \sin\left(\frac{b}{a}\ln\left(\frac{as}{\sqrt{2a^2+b^2}} + 1\right)\right), 1 \right\rangle$

the parametrization by arclength

$L = \int_0^1 |r'(t)|\, dt = \int_0^1 e^{at}\sqrt{2a^2+b^2}\, dt = \sqrt{2a^2+b^2} \cdot \int_0^1 e^{at}\, dt$

$= \sqrt{2a^2+b^2} \cdot \left[\frac{1}{a}e^{at}\right]_0^1 = \frac{\sqrt{2a^2+b^2}}{a}(e^a - 1)$

length of the curve

b) $k = \left|\frac{dT}{ds}\right| = \frac{|T'(t)|}{|r'(t)|}$ $\quad T(t) = \frac{r'(t)}{|r'(t)|}$

10

$r'(t) = \langle ae^{at}\cos(bt) - be^{at}\sin(bt),\ ae^{at}\sin(bt) + be^{at}\cos(bt),\ ae^{at} \rangle$

$|r'(t)| = e^{at}\sqrt{2a^2+b^2}$

$T(t) = \frac{1}{\sqrt{2a^2+b^2}} \cdot \langle a\cos(bt) - b\sin(bt),\ a\sin(bt) + b\cos(bt),\ a \rangle$

$T'(t) = \frac{1}{\sqrt{2a^2+b^2}} \cdot \langle -ab\sin(bt) - b^2\cos(bt),\ ab\cos(bt) - b^2\sin(bt),\ 0 \rangle$

$|T'(t)| = \frac{1}{\sqrt{2a^2+b^2}} \cdot \sqrt{(-ab\sin(bt) - b^2\cos(bt))^2 + (ab\cos(bt) - b^2\sin(bt))^2}$

$= \frac{1}{\sqrt{2a^2+b^2}} \cdot \sqrt{a^2b^2\sin^2(bt) + b^4\cos^2(bt) + 2ab^3\sin(bt)\cos(bt) + a^2b^2\cos^2(bt) + b^4\sin^2(bt) - 2ab^3\cos(bt)\sin(bt)}$

$= \frac{1}{\sqrt{2a^2+b^2}} \cdot \left(a^2b^2(\sin^2(bt) + \cos^2(bt)) + b^4(\cos^2(bt) + \sin^2(bt))\right)^{0.5}$

$= \frac{1}{\sqrt{2a^2+b^2}} \cdot \sqrt{a^2b^2+b^4}$

$\underline{k} = \frac{\frac{1}{\sqrt{2a^2+b^2}}}{e^{at}\sqrt{2a^2+b^2}} \cdot \sqrt{b^2(a^2+b^2)} = \frac{1}{e^{at}(2a^2+b^2)} \cdot b\sqrt{a^2+b^2}$

← curvature

3 a) tangent plane :

$0 = f_x(x_0\, y_0\, z_0)(x-x_0) + f_y(x_0\, y_0\, z_0)(y-y_0) + f_z(x_0\, y_0\, z_0)(z-z_0)$

$= 2\cdot(x_0)(x-x_0) + \frac{1}{2}(y_0)(y-y_0) + \frac{2}{9}(z_0)(z-z_0)$

$= 2\cdot(x-1) + 1\cdot(y-2) + \frac{2}{3}(z-3)$

$= 2x - 2 + y - 2 + \frac{2}{3}z - 2$

$\Longrightarrow 2x + y + \frac{2}{3}z = 6$ 5/5

b)

As we saw at a) $f_z(x_0 y_0) = \frac{2}{3} \neq 0 \rightarrow$ we can use the implicit function th^m. s.t. :

$f_x(x_0 y_0) = -\frac{\frac{x_0}{y_0}}{\frac{3}{2}} = -\frac{1}{2}\cdot\frac{2}{3} = -3$

$f_y(x_0 y_0) = -(\underbrace{\quad}_{\partial f/\partial y})/(3/2) = -(9/4)/(3/2) = -\frac{3}{2}$

10/10

Linearization of f at $(x_0\, y_0)$ gives :

$z = L(x_0, y_0) = f(x_0\, y_0) + f_x(x_0\, y_0)(x-x_0) + f_y(x_0\, y_0)(y-y_0)$

$= 3 - 3(x-1) - \frac{3}{2}(y-2)$

$\Longrightarrow z = 3 - 3x + 3 - \frac{3}{2}y + 3 = 9 - 3x - \frac{3}{2}y \Longrightarrow$

$z + 3x + \frac{3}{2}y = 9 \Longrightarrow \frac{2}{3}z + 2x + y = 6$

which agrees with the tangent plane found at a).

Implicit function th^m : $\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} = f_x$

$\frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z} = f_y$

c) $g(x\,y\,z) = x^2 + y^2 + z^2$ & $f(x\,y\,z) = \frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{9} = 3$

$\cancel{\lambda \nabla f = \nabla g \iff \begin{cases} \lambda f_x = g_x \\ \lambda f_y = g_y \\ \lambda f_z = g_z \\ x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 3 \end{cases} \iff \begin{cases} \lambda \cdot 2x = 2x \\ \lambda \cdot \frac{1}{2} y = 2y \end{cases}}$

$$\nabla f = \lambda \cdot \nabla g \iff \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 3 \end{cases} \iff \begin{cases} 2x = \lambda 2x \\ \frac{1}{2} y = \lambda 2y \\ \frac{2}{9} z = \lambda 2z \\ x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 3 \end{cases} \iff$$

$\lambda = \frac{1}{9}$

$\lambda = \frac{1}{4}$

$\lambda = 1$

$$\begin{cases} \lambda = 1 \text{ or } x = 0 \\ \lambda = \frac{1}{4} \text{ or } y = 0 \\ \lambda = \frac{1}{9} \text{ or } z = 0 \\ x^2 + y^2/4 + \frac{z^2}{9} = 3 \end{cases} \iff \begin{cases} x = y = 0 \to z = \pm\sqrt{27} \\ x = z = 0 \to y = \pm\sqrt{12} \\ y = z = 0 \to x = \pm\sqrt{3} \\ \cancel{x^2 + y^2/4 + \frac{z^2}{9} = 3} \end{cases}$$

points P_1, P_2, P_3 are:

10/10

$P_1 = (0, 0, \pm\sqrt{27})$

$P_2 = (0, \pm\sqrt{12}, 0)$

$P_3 = (\pm\sqrt{3}, 0, 0)$

$g(P_1) = 27 \leftarrow$ a maximum for $g(x\,y\,z)$

$g(P_2) = 12$

$g(P_3) = 3 \leftarrow$ ~~closest to the~~ a minimum for $g(x\,y\,z)$

Thus, the points farthest away from the origin ~~is~~:

$(0, 0, \pm\sqrt{27})$

points closest to the origin: $(\pm\sqrt{3}, 0, 0)$

(4) $x = r\cos(t)$ $y = r\sin(t)$ $z = z$

$x^2 + y^2 = r^2$ $0 \le \theta \le 2\pi$ $0 \le r \le 1$

20/20

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{0}^{4-x^2-y^2} e^{x^2+y^2+z}\, dz\, dx\, dy$$ (since $x \le \sqrt{1-y^2} \Rightarrow x^2 \le 1-y^2 \Rightarrow x^2+y^2 \le 1 = r^2$)

$$= \int_0^1\int_0^{2\pi}\int_0^{4-r^2} e^{r^2+z} \cdot r\, dz\, d\theta\, dr$$

$$= \int_0^1\int_0^{2\pi}\int_0^{4-r^2} r e^{r^2+z}\, dz\, d\theta\, dr$$

$$= \int_0^1\int_0^{2\pi} \left[r e^{r^2+z}\right]_0^{4-r^2} d\theta\, dr$$

$$= \int_0^1\int_0^{2\pi} r\left(e^{r^2+4-r^2} - e^{r^2}\right) d\theta\, dr$$

$$= 2\pi \int_0^1 re^4 - re^{r^2}\, dr$$

$$= 2\pi\left(\left[\tfrac{1}{2}r^2e^4\right]_0^1 - \int_0^1 re^{r^2}\, dr\right)$$

set $u = r^2$ & $\frac{du}{dr} = 2r$ then

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{0}^{4-x^2-y^2} e^{x^2+y^2+z}\, dz\, dx\, dy$$

$$= 2\pi\left(\left(\tfrac{1}{2}e^4\right) - \int_0^1 \tfrac{1}{2}e^u\, du\right)$$

$$= \pi e^4 - 2\pi \cdot \tfrac{1}{2}\int_0^1 e^u\, du$$

$$= \pi\left(e^4 - \left[e^u\right]_0^1\right)$$

$$= \pi\left(e^4 - e^1 + 1\right)$$

$$= \pi\left(e^4 - e + 1\right)$$